in which

$$k = \frac{r_e}{r_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

is the ratio of the external radius to the internal radius of the cylinder. By substituting Eqs. 1, 2, and 3 in the corresponding criterion, the desired conditions of elastic loading are obtained.

ELASTIC LOADING FOR VON MISES' CRITERION

The substitution of Eqs. 1, 2, and 3 in the relation of Von Mises, given by

$$\left(\sigma_r - \sigma_\theta\right)^2 + \left(\sigma_\theta - \sigma_z\right)^2 + \left(\sigma_z - \sigma_r\right)^2 < 2\,\sigma_0{}^2 \quad \ldots\ldots \quad (5)$$

yields the condition of elastic loading corresponding to this criterion

$$\frac{2}{\left(k^2 - 1\right)^2}\left[3\,k^4\,\frac{r_i^4}{r^4}\left(p_i - p_e\right)^2 + p_i^2 + k^4\,p_e^2 - 2\,k^2\,p_i\,p_e + p_l^2\left(k^2 - 1\right)^2 + 2\,p_i\,p_l\left(k^2 - 1\right) - 2\,p_e\,p_l\,k^2\left(k^2 - 1\right)\right] < 2\,\sigma_0^2 \quad \ldots \quad (6)$$

in which σ_0 is the elastic limit of the material for pure tension. The left side of Eq. 6 is a maximum for $r = r_i$, and demonstrates that plastic deformations will occur, either first at the internal diameter of the cylinder whatever the relative values of p_i and p_e, or simultaneously in the entire thickness of the cylinder for the particular case $p_i = p_e = 0$ and $p_l = \pm\,\sigma_0$.

Now, consider the case in which a plastic deformation is possible. The relation, $r = r_i$ is written, the inequality[7] in Eq. 6 becomes equality. On the pressure space p_i, p_l, p_e, the surface described by this equality is an elliptic cylinder[7] with its axis pointing in the direction (1, 1, 1). The elliptic cross-section varies both in dimension and orientation, with k. This surface has meaning only as long as p_i and p_e are positive, while

$$p_l = -\,\frac{L}{\pi\left(r_e^2 - r_i^2\right)} \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

may be positive, negative or vanishing, depending on the value and sign (tension or compression) of the longitudinal load L.

This surface is studied in the system of orthonormal axes V, W, Z, with V and W being respectively coincident with the minor and major axes of the ellipse

[7] Epain, R., "Contribution à l'étude de la résistance des cylindres épais elasto-plastiques," thesis presented to the University of Paris, at Paris, France, in 1961, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

of the normal cross section, while Z, parallel to the generating line of the elliptic cylinder, is inclined at equal angles to p_i, p_l, p_e.

The dimensions of the ellipse of the normal cross section are given by

$$\frac{\text{minor axis}}{2} = \frac{\sigma_0 (M - 1)}{\sqrt{4 M^2 - M + 1 + \sqrt{7 M^4 + 10 M^3 - 2 M + 1}}} \quad \ldots \quad (8)$$

and

$$\frac{\text{major axis}}{2} = \frac{\sigma_0 (M - 1)}{\sqrt{4 M^2 - M + 1 - \sqrt{7 M^4 + 10 M^3 - 2 M + 1}}} \quad \ldots \quad (9)$$

while its orientation, relative to the projections p'_i, p'_l, p'_e of the axes p_i, p_l, p_e onto the plane, π, perpendicular to Z, is determined by

$$\tan \psi = \sqrt{\frac{\left(\frac{V_1}{V_3}\right)^2 + \left(\frac{V_2}{V_3}\right)^2 + 1}{\left(\frac{W_1}{W_3}\right)^2 + \left(\frac{W_2}{W_3}\right)^2 + 1}} \quad \ldots\ldots\ldots \quad (10a)$$

in which

$$\psi = \text{angle } V, p'_e \quad \ldots\ldots\ldots\ldots\ldots \quad (10b)$$

$$\frac{V_1}{V_3} = \frac{3 M^2 (M - 1) + \chi_1}{3 M^2 (M - 1) - M \chi_1} \quad \ldots\ldots\ldots\ldots \quad (10c)$$

$$\frac{W_1}{W_3} = \frac{3 M^2 (M - 1) + \chi_2}{3 M^2 (M - 1) - M \chi_2} \quad \ldots\ldots\ldots\ldots \quad (10d)$$

$$\frac{V_2}{V_3} = \frac{3 M^2 (M - 1)^2 + (M - 1) \chi_1}{3 M^2 (M - 1)^2 - (3 M + 1) M \chi_1} \quad \ldots\ldots\ldots \quad (10e)$$